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FORCED SUBHARMONIC OSCILLATIONS OF THE SIMPLE PENDULUM*

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Symmetric subharmonic solutions of second order differential equation defining oscillations of the simple pendulum subjected to an external sinusoidal force are considered. In the case of small amplitude of the exciting force the subharmonic solutions are analytically determined and then continued in the domain of large amplitudes of that force, using numerical computations. Branching of derived solutions is investigated.

1. Introduction. Let us consider the differential equation

$$x^{\cdot\cdot} + \mu \sin x = e \sin t \tag{1.1}$$

where x is the unknown function, t is the independent variable, and e and μ are parameters. Periodic solutions of this equation which for $e = 0$ is the same as the periodic solutions of the respective homogeneous equation were sought in /1/, where the obtained solutions were denoted by $x_z(t, e)$, $x_{n/m}^{(r)}(t, e)$, $\bar{x}_{n/m}^{(r)}(t, e)$. The solution $x_z(t, e)$ is of period 2π, exists when $|e| \ll 1$, $\mu \neq n^2$ $(n = 1, 3, 5, \ldots)$, and satisfies the condition $x_z(t, 0) = 0$. In $x_{n/m}^{(r)}(t, e)$, $\bar{x}_{n/m}^{(r)}(t, e)$ m and n are relatively prime positive integers, with one of these numbers in $x_{n/m}^{(r)}$ is even, $r = 0, 1, \ldots, 2m - 1$. Solutions $x_{n/m}^{(r)}$, $\bar{x}_{n/m}^{(r)}$ are of period $2\pi m$, are determinate when $|e| \ll 1$, $\mu > n^2/m^2$, and coincide with $2\pi m/n$-periodic solutions of the homogeneous equation when $e = 0$. Solutions x_z, $x_{n/m}^{(r)}$, $\bar{x}_{n/m}^{(r)}$ depend analytically on e in the neighborhood of point $e = 0$.

The numerical derivation of solution x_z reduces to the solution for Eq.(1.1) of the boundary value problem

$$x(0) = x^{\cdot}\left(\frac{\pi}{2}\right) = 0 \tag{1.2}$$

The derivation of solutions $x_{n/m}^{(r)}$ reduces to solving the boundary value problem

$$x(0) = x^{\cdot}\left(\frac{\pi m}{2}\right) = 0 \tag{1.3}$$

when m and n are odd numbers, and when one of these is even, to solving the boundary value problem

$$x(0) = x(\pi m) = 0 \tag{1.4}$$

The derivation of solutions $\bar{x}_{n/m}^{(r)}$ reduces to solving the boundary value problem

$$x^{\cdot}\left(\frac{\pi}{2}\right) = x^{\cdot}\left(\frac{\pi}{2} + \pi m\right) = 0 \tag{1.5}$$

The solution of the boundary value problem can be defined as the surface S in space $R^3(x^{\cdot}(0), e, \mu)$ or as surface S' in space $R^3(x(\pi/2), e, \mu)$. Surfaces S and S' are diffeomorphic. We define the solutions of boundary value problems (1.3) — (1.5) as surfaces $S_{n/m} \subset R^3(x^{\cdot}(0), e, \mu)$ and $S_{n/m}' \subset R^3(x(\pi/2), e, \mu)$, respectively. When n is odd, then $S_{n/1} \subset S$. Properties of surfaces S, $S_{3/1}$, $S_{3/1}'$ were investigated in /1/, where surfaces S', $S_{3/1}$, $S_{3/1}'$ were denoted, respectively, by $\bar{S}$, S', $\bar{S}'$. Here we consider surfaces $S_{n/m}$ and $S_{n/m}'$ with $m > 1$. These surfaces define the dependence of symmetric subharmonic forced oscillations of the simple pendulum on parameters e and μ. According to /1/ it is sufficient to investigate these surfaces for $e \geqslant 0$.

2. Resonance curves. Let $x(t)$ be a solution of the boundary value problem (1.1), (1.2). Then /1/

$$x(-t) = -x(t), \quad x(t + \pi) = -x(t) \tag{2.1}$$

Consider the respective equation in variations

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$$y^{\cdot\cdot} + \mu y \cos x(t) = 0 \tag{2.2}$$

where by virtue of (2.1) $\cos x(t)$ is an even π-periodic function. Let $y_1(t)$, $y_2(t)$ be solutions of (2.2) with initial conditions $y_1(0) = y_2^{\cdot}(0) = 1$, $y_1^{\cdot}(0) = y_2(0) = 0$. Then $y_1(t)$ is an even, and $y_2(t)$ an odd function. Because of this and the relation (Liouville's theorem)

$$y_1\left(\frac{\pi}{2}\right) y_2^{\cdot}\left(\frac{\pi}{2}\right) - y_1^{\cdot}\left(\frac{\pi}{2}\right) y_2\left(\frac{\pi}{2}\right) = 1 \tag{2.3}$$

the monodromy matrix of Eq.(2.2)

$$C = \begin{Vmatrix} y_1(\pi) & y_2(\pi) \\ y_1^{\cdot}(\pi) & y_2^{\cdot}(\pi) \end{Vmatrix}$$

can be represented as follows:

$$C = \begin{Vmatrix} y_1\left(\frac{\pi}{2}\right) y_2^{\cdot}\left(\frac{\pi}{2}\right) + y_1^{\cdot}\left(\frac{\pi}{2}\right) y_2\left(\frac{\pi}{2}\right) & 2y_2\left(\frac{\pi}{2}\right) y_2^{\cdot}\left(\frac{\pi}{2}\right) \\ 2y_1\left(\frac{\pi}{2}\right) y_1^{\cdot}\left(\frac{\pi}{2}\right) & y_1\left(\frac{\pi}{2}\right) y_2^{\cdot}\left(\frac{\pi}{2}\right) + y_1^{\cdot}\left(\frac{\pi}{2}\right) y_2\left(\frac{\pi}{2}\right) \end{Vmatrix}$$

The characteristic equation for (2.2) is of the form $\rho^2 - 2A\rho + 1 = 0$, where

$$A = y_1\left(\frac{\pi}{2}\right) y_2^{\cdot}\left(\frac{\pi}{2}\right) + y_1^{\cdot}\left(\frac{\pi}{2}\right) y_2\left(\frac{\pi}{2}\right) \tag{2.4}$$

When $|A| < 1$, the solution of $x(t)$ is stable in the linear approximation, while when $|A| > 1$ it is unstable. Boundaries of the stability domain are curves on surface S determined by the equations $A = 1$ and $A = -1$. When $A = 1$ we have by virtue of relations (2.3) and (2.4) $y_1^{\cdot}(\pi/2) y_2(\pi/2) = 0$. If $y_1^{\cdot}(\pi/2) = 0$, function $y_1(t)$ is π-periodic, while when $y_2(\pi/2) = 0$, function $y_2(t)$ has that property. Hence curves $A = 1$ can be of two types, viz. Γ_1^+ and Γ_1^-. Along the curve Γ_1^+ (Γ_1^-) Eq.(2.2) has an even (odd) π-periodic solution. Similarly, when $A = -1$, we have $y_1(\pi/2) y_2^{\cdot}(\pi/2) = 0$. If $y_1(\pi/2) = 0$, function $y_1(t)$ is π-antiperiodic (and, consequently, 2π-periodic); if $y_2^{\cdot}(\pi/2) = 0$, function $y_2(t)$ has that property. Curves $A = -1$ can be of two types, viz. Γ_2^+ and Γ_2^-. On curve Γ_2^+ (Γ_2^-) Eq.(2.2) has an even (odd) π-antiperiodic solution. The results of computations of curves $\Gamma_m^{\pm}$ ($m = 1, 2$) appear in /1/.

Let $|A| < 1$. We determine ω using the condition $A = \cos \pi\omega$. Then by virtue of relations (2.3) and (2.4) it is possible to represent matrix C as (cf. /2/)

$$C = \begin{Vmatrix} \cos \pi\omega & \frac{1}{a} \sin \pi\omega \\ -a \sin \pi\omega & \cos \pi\omega \end{Vmatrix}$$

where $a \neq 0$ is some number. Consider solutions $u_1(t) = y_1(t)$, $u_2(t) = a y_2(t)$ of Eq.(2.2). It can be shown that

$$u_1(t) = \psi_1(t) \cos \omega t - \psi_2(t) \sin \omega t, \qquad u_2(t) = \psi_1(t) \sin \omega t + \psi_2(t) \cos \omega t \tag{2.5}$$

where $\psi_1(t)$ is an even and $\psi_2(t)$ an odd π-periodic function.

Assume that m is a positive integer and consider curves $\Gamma_m \subset S$ on which Eq.(2.2) has nontrivial periodic solutions with the lowest period πm. Such curves are called resonance curves of the m-th order. When $m = 1, 2$, then $\Gamma_m = \Gamma_m^+ \cup \Gamma_m^-$. When $m > 2$, the resonance curves are determined by formulas $\cos \pi m\omega = 1$, $\cos \pi k\omega \neq 1$ ($k = 1, \ldots, m-1$), by virtue of which $\omega = 2n/m$, where n is an integer relatively prime to m. It follows from (2.5) that all solutions of Eq.(2.2) along curves Γ_m are πm-periodic when $m > 2$.

Numerical computation of curves Γ_m for $m > 2$ reduces to solving the boundary value problem (1.2) and $A = \cos 2\pi n/m$ for the system consisting of Eq.(1.1) and two Eqs.(2.2), with A determined by formula (2.4). There exist other methods of constructing resonance curves, for example, the boundary value problem (1.2) $y(0) = y(\pi m/2) = 0$, can be solved for system (1.1), (2.2). However the method described here requires less computer time.

We denote the projection of curve Γ_m on the (e, μ)-plane by γ_m. Some of the γ_m curves are shown in Fig.1.

3. Periodic solutions of the second kind. According to /2/ curves γ_m can be branching curves πm-periodic (when m is even) and $2\pi m$-periodic (when m is odd) of solutions of Eq.(1.1). When $m > 2$ such solutions are called periodic solutions of the second kind /2/. For the investigation of these solutions we carry out the following transformations. Let $(e_*, \mu_*) \in \gamma_m$ and $x_*(t)$ be the solution of the boundary value problem (1.2), $(x_*^{\cdot}(0), e_*, \mu_*) \in \Gamma_m$, corresponding to point (e_*, μ_*). Then, using the notation

$$q = x - x_*(t), \quad \varepsilon = e - e_*, \quad \delta = \mu - \mu_*, \; f(t) = \mu_* \cos x_*(t)$$
$$H(q, t, \varepsilon, \delta) = \varepsilon \sin t + \mu_* \sin x_*(t) + f(t) q - (\mu_* + \delta) \sin [x_*(t) + q]$$

we can write Eq.(1.1) as

$$q^{\cdot\cdot} + f(t) q = H(q, t, \varepsilon, \delta) \tag{3.1}$$

where

$$f(t + \pi) = f(t), \quad H(q, t + \pi, \varepsilon, \delta) = -H(-q, t, \varepsilon, \delta) \tag{3.2}$$

$$f(t) = f(-t), \quad H(q, t, \varepsilon, \delta) = -H(-q, -t, \varepsilon, \delta) \tag{3.3}$$

Function $H(q, t, \varepsilon, \delta)$ is analytic with respect to q, ε, δ at point $q = \varepsilon = \delta = 0$, and $H(q, t, \varepsilon, \delta) = O(q^2 + |\varepsilon| + |\delta|)$. The investigation of peroidic solutions of Eq.(1.1) that coincide with the solution $x_*(t)$ when $e = e_*$, $\mu = \mu_*$ is equivalent to the investigation of periodic solutions of Eq.(3.1) which vanish for $\varepsilon = \delta = 0$.

Equation (3.1) on curves Γ_1 was considered in /1/. The investigation carried out there made possible a more precise determination of surfaces S, $S_{2/1}$ and S', $S_{2/1}'$. Analysis of Eq.(3.1) on curves Γ_2^- (curves Γ_2^+ were not disclosed) provides the possibility of investigating surface S. Such investigation yields results that are similar to those in /3/. Below, we consider Eq.(3.1) along curves Γ_m when $m > 2$.

Let $m > 2$. be an integer. Let us investigate branching of the $2\pi m$-periodic solutions of Eq.(3.1) along curves γ_{2m} and γ_m (curves γ_m are considered only for odd m). Along these curves the linearly independent solutions of equation $q^{\cdot\cdot} + f(t) q = 0$ (cf. (2.2)) can be taken in the form (2.5), where $\omega = n/m$ and n is a positive integer relatively prime to m. Along curves γ_m the number n must be even. Consider the auxilliary system

$$q^{\cdot\cdot} + f(t) q = H(q, t, \varepsilon, \delta) - p_1 u_1(t) - p_2 u_2(t)$$
$$\int_0^{2\pi m} q u_1(t)\, dt = a_1, \quad \int_0^{2\pi m} q u_2(t)\, dt = a_2$$

where q is the unknown function, p_1 and p_2 are unknown constants, and a_1 and a_2 are arbitrary constants. When $|a_1|$, $|a_2|$, $|\varepsilon|$, $|\delta|$ are fairly small, this system has a unique $2\pi m$-periodic in t solution /4,5/

$$q = q_*(t, a_1, a_2, \varepsilon, \delta), \quad p_j = p_j^*(a_1, a_2, \varepsilon, \delta) \quad (j = 1, 2) \tag{3.4}$$

which analytically depends on $a_1, a_2, \varepsilon, \delta$ and satisfies the conditions $q_*(t, 0, 0, 0, 0) = 0$, $p_j^*(0, 0, 0, 0) = 0$. The derivation of $2\pi m$-periodic solutions of Eq. (3.1) that vanish when $\varepsilon = \delta = 0$ is equivalent to the determination of roots $a_j = a_j(\varepsilon, \delta)$ $(j = 1, 2)$ of system

$$p_1^*(a_1, a_2, \varepsilon, \delta) = 0, \quad p_2^*(a_1, a_2, \varepsilon, \delta) = 0 \tag{3.5}$$

such that $a_j(0, 0) = 0$. Let $a_j(\varepsilon, \delta)$ be the roots of that system, and $a_j(0, 0) = 0$. Then $q = q_*[t, a_1(\varepsilon, \delta), a_2(\varepsilon, \delta), \varepsilon, \delta]$ is a $2\pi m$-periodic solution of Eq.(3.1), whose characteristic indices λ are of the form

$$\lambda^2 = -\frac{M^4}{(2\pi m W)^2} \frac{\partial(p_1^*, p_2^*)}{\partial(a_1, a_2)} [a_1(\varepsilon, \delta), a_2(\varepsilon, \delta), \varepsilon, \delta](1 + o(1))$$
$$M = \int_0^{2\pi m} u_1^2(t)\, dt = \int_0^{2\pi m} u_2^2(t)\, dt = m \int_0^{\pi} [\psi_1^2(t) + \psi_2^2(t)]\, dt$$
$$W = u_1 u_2^{\cdot} - u_1^{\cdot} u_2 = \mathrm{const}$$

where W is the Wronskian of functions (2.5), and $o(1)$ denotes the function of ε and δ which approaches zero as $\varepsilon \to 0$, $\delta \to 0$.

Let us point out some of the properties of solution (3.4). Using formulas (3.3) and the evenness of functions $u_1(t)$ and $u_2(t)$ it is possible to prove that

$$-q_* (-t, a_1, a_2, \varepsilon, \delta) = q_* (t, -a_1, a_2, \varepsilon, \delta) \tag{3.6}$$
$$p_1^* (-a_1, a_2, \varepsilon, \delta) = -p_1^* (a_1, a_2, \varepsilon, \delta), \quad p_2^* (-a_1, a_2, \varepsilon, \delta) = p_2^* (a_1, a_2, \varepsilon, \delta)$$

We introduce the notation

$$Q = -\left\| \begin{matrix} \cos \pi\omega & \sin \pi\omega \\ -\sin \pi\omega & \cos \pi\omega \end{matrix} \right\|, \quad a = (a_1, a_2)^T, \quad \beta = (\varepsilon, \delta)^T$$
$$p_* (a, \beta) = (p_1^* (a_1, a_2, \varepsilon, \delta), \ p_2^* (a_1, a_2, \varepsilon, \delta))^T$$
$$q_* (t, a, \beta) = q_* (t, a_1, a_2, \varepsilon, \delta)$$

By virtue of (2.5) and (3.2) for any integral k we have

$$q_* (t + \pi k, \ a, \beta) = (-1)^k q_* (t, Q^k a, \beta), \quad p_* (Q_a^k, \beta) = Q^k p_* (a, \beta) \tag{3.7}$$

We pass to the investigation of system (3.5). It follows from (3.7) that $Qp_* (0, \beta) = p_* (0, \beta)$ and, since unity is not an eigenvalue of matrix Q, $p_* (0, \beta) = 0$. Thus $q_* (t, 0, \beta)$ (cf. (3.6)) is an odd π-antiperiodic (cf. (3.7)) solution of Eq.(3.1), which satisfies the condition $(x_*^{\cdot} (0) + q_*^{\cdot} (0, 0, \beta), e_* + \varepsilon, \mu_* + \delta) \in S$. The investigated curve γ_m or γ_{2m} in the (ε, δ)-plane is defined by the equation

$$\frac{\partial (p_1^*, p_2^*)}{\partial (a_1, a_2)} (0, 0, \varepsilon, \delta) = 0$$

Let $b (\varepsilon, \delta)$ be a root of the equation

$$p_2^* (0, b, \varepsilon, \delta) = 0 \tag{3.8}$$

which is not identically zero and satisfies the condition $b (0, 0) = 0$. We introduce the notations: $a^{(0)} (\beta) = (0, \ b (\varepsilon, \ \delta))^T$, $a^{(k)} (\beta) = Q^k a^{(0)} (\beta)$ $(k = 1, 2, \ldots)$. Then, by virtue of (3.6) and (3.7) $p_* [a^{(k)} (\beta), \beta] = 0$. Matrix Q^{2m} is an unit matrix, hence among vectors $a^{(k)} (\beta)$ $(k = 0, 1, \ldots)$ there are not more than $2m$ different ones. If $a^{(k)} (\beta) = a^{(l)} (\beta)$, then $\cos [\pi (m + n) (k - l)/m] = 1$ or, what is the same, $(m + n) (k - l) \equiv 0 \pmod{2m}$. Since m and n are relatively prime integers, $m + n$ and m are also relatively prime. Consequently $k - l = rm$, where $r (m + n) \equiv 0 \pmod 2$.

Let m and n be odd. Then $m + n$ is even and $k \equiv l \pmod m$. In this case there are only m different (for instance, $a^{(2k)} (\beta)$, $k = 0, 1, \ldots, m - 1$) vectors among $a^{(k)} (\beta)$ $(k = 0, 1, \ldots)$ If one of the numbers m and n is even (the other is odd by virtue of primality of m and n then $m + n$ is odd and $k \equiv l \pmod{2m}$. In this case there are $2m$ different vectors (for instance, $a^{(k)} (\beta)$, $k = 0, 1, \ldots, 2m - 1$) among $a^{(k)} (\beta)$ $(k = 0, 1, \ldots)$. The first of these cases is only possible on curves γ_{2m}, while the second is possible on curves γ_{2m} (m even, n odd), as well as on curves γ_m (m odd, n even).

The $2\pi m$-periodic solutions of Eq.(1.1), $x^{(k)} (t) = x_* (t) + q_* [t, a^{(k)} (\beta), \beta]$ $(k = 0, 1, \ldots)$ correspond to roots $a^{(k)} (\beta)$ of system (3.5). Solution $x^{(0)} (t)$ is odd and satisfies boundary conditions (1.4). The remaining solutions are related to $x^{(0)} (t)$ by formula $x^{(k)} (t) = (-1)^k x^{(0)} (t + \pi k)$. If either m or n is even, then among solutions $x^{(k)} (t)$ $(k = 0, 1, \ldots)$ $2m$ are different, when both are odd, there are only m different solutions. In the latter case solutions $x^{(k)} (t)$ $(k = 0, 1, \ldots)$ are πm-antiperiodic and solution $x^{(0)} (t)$ satisfies the boundary conditions (1.3).

Let us consider in detail the case when one of the numbers m and n is even. Then $(-Q^m)$ is an unit matrix, and when $k = m$ the second of formulas (3.7) assumes the form $p_* (-a, \beta) = -p_* (a, \beta)$. Comparing this formula with (3.6) we find that

$$p_j^* (a_1, a_2, \varepsilon, \delta) = a_j \varphi_j (a_1^2, a_2^2, \varepsilon, \delta) \quad (j = 1, 2) \tag{3.9}$$

where $\varphi_j (z_1, z_2, \varepsilon, \delta)$ are analytic functions of $z_1, z_2, \varepsilon, \delta$ at point $z_1 = z_2 = \varepsilon = \delta = 0$, and $\varphi_j (0, 0, 0, 0) = 0$. When $b \neq 0$, Eq.(3.8) assumes by virtue of (3.9) the form

$$\varphi_2 (0, b^2, \varepsilon, \delta) = 0 \tag{3.10}$$

Consider now the equation

$$\varphi_1 (b^2, 0, \varepsilon, \delta) = 0 \tag{3.11}$$

Let $b (\varepsilon, \delta)$ be its root that is not identically zero and satisfies the condition $b (0, 0) = 0$. We introduce vectors $\bar{a}^{(0)} (\beta) = (b (\varepsilon, \delta), 0)^T$, $\bar{a}^{(k)} (\beta) = Q^k \bar{a}^{(0)} (\beta)$ $(k = 1, 2, \ldots)$. It is possible to show that $p_* [\bar{a}^{(k)} (\beta), \beta] = 0$ and that among vectors $\bar{a}^{(k)} (\beta)$ $(k = 0, 1, \ldots, 2m - 1)$

none are the same. To roots $\bar{a}^{(k)}(\beta)$ of system (3.5) correspond $2\pi m$-periodic solutions of Eq.(1.1), $x^{(k)}(t) = x_*(t) + q_*[t, \bar{a}^{(k)}(\beta), \beta]$ $(k = 0, 1, \ldots)$. All these solutions can be expressed in terms of $x^{(0)}(t)$ by using formula $x^{(k)}(t) = (-1)^k x^{(0)}(t + \pi k)$, and there are only $2m$ different ones among them.

If n is even, the totality of solutions $\{x^{(k)}(t)$: $k = 0, 1, \ldots\}$ and $\{x^{(k)}(t) : k = 0, 1, \ldots\}$ do not intersect for any selection of Eqs.(3.10) and (3.11) that generate these roots. Solution $x^{(s)}(t)$, where $s \equiv (m-1)/2 \pmod{2m}$, satisfies the boundary conditions (1.5).

When m is even

$$Q^{m/2} = \pm \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \tag{3.12}$$

from which by virtue of formulas (3.7) with $k = m/2$ and (3.9) we have

$$\varphi_1(z_1, z_2, \varepsilon, \delta) = \varphi_2(z_2, z_1, \varepsilon, \delta) \tag{3.13}$$

Hence Eqs.(3.10) and (3.11) are equivalent. Let $b(\varepsilon, \delta)$ be a root of these equations such that $b(0, 0) = 0$. Then by virtue of (3.12) the set of solutions $\{x^{(k)}(t)$: $k = 0, 1, \ldots\}$ and $\{x^{(k)}(t)$: $k = 0, 1, \ldots\}$ that correspond to that root are the same.

Consider the equation

$$\varphi_2(b_*^2, b_*^2, \varepsilon, \delta) = 0 \tag{3.14}$$

when m is even.

Let $b_*(\varepsilon, \delta)$ be its root that is not identically zero and satisfies the condition $b_*(0, 0) = 0$, $a_*^{(0)}(\beta) = (b_*(\varepsilon, \delta), \; b_*(\varepsilon, \delta))^T$, $a_*^{(k)}(\beta) = Q^k a_*^{(0)}(\beta)$ $(k = 1, 2, \ldots)$. Then $p_*[a_*^{(k)}(\beta), \beta] = 0$, and among vectors $a_*^{(k)}(\beta)$ $(k = 0, 1, \ldots)$ $2m$ are different. To these vectors correspond $2\pi m$-periodic solutions of Eq.(1.1) $x_*^{(k)}(t) = x_*(t) + q_*[t, a_*^{(k)}(\beta), \beta]$ $(k = 0, 1, \ldots)$ for which $x_*^{(k)}(t) = (-1)^k x_*^{(0)}(t + \pi k)$.

If $m \equiv 2 \pmod 4$, the sets $\{x^{(k)}(t)$: $k = 0, 1, \ldots\}$ and $\{x_*^{(k)}(t) : k = 0, 1, \ldots\}$ are different for any selection of Eqs. (3.10) and (3.14) which generate their roots. Solution $x_*^{(s)}(t)$, where

$$s \equiv (-1)^{(n-1)/2}\, m/4 - {}^1/_2 \pmod{2m}$$

satisfies the boundary conditions (1.5).

When $m \equiv 0 \pmod 4$, then $Q^{3m/4}(0, b)^T = (b_*, b_*)^T$ for any b and b_* linked by the relation $b^2 = 2b_*^2$. Thus, when $b_*(\varepsilon, \delta)$ is the root of Eq.(3.14), $b(\varepsilon, \delta) = b_*(\varepsilon, \delta)\sqrt{2}$ is by virtue of (3.7), (3.9), and (3.13), the root of Eqs.(3.10) and (3.11). The sets of solutions $\{x^{(k)}(t) : k = 0, 1, \ldots\}$ and $\{x_*^{(k)}(t) : k = 0, 1, \ldots\}$ which correspond to these roots are the same.

We point out one more property of functions φ_1 and φ_2, which will be applied below. Using (3.7) it is possible to prove that when mn is even and $m \geqslant 3$

$$\varphi_1(0, 0, \varepsilon, \delta) = \varphi_2(0, 0, \varepsilon, \delta) \tag{3.15}$$

$$\frac{\partial \varphi_l(0, 0, \varepsilon, \delta)}{\partial z_j} = \frac{\partial \varphi_s(0, 0, \varepsilon, \delta)}{\partial z_k} \quad (j, k, s, l = 1, 2)$$

4. 4π-periodic solutions of the second kind.

Let us consider curves γ_4. Here $m = 2, n = 1, 3, 5 \ldots$. We represent functions φ_1 and φ_2 in (3.9) as (cf. (3.13))

$$\varphi_1(z_1, z_2, \varepsilon, \delta) = C_1 z_1 + C_2 z_2 + A\varepsilon + B\delta + O(z_1^2 + z_2^2 + \varepsilon^2 + \delta^2) \tag{4.1}$$

$$\varphi_2(z_1, z_2, \varepsilon, \delta) = C_2 z_1 + C_1 z_2 + A\varepsilon + B\delta + O(z_1^2 + z_2^2 + \varepsilon^2 + \delta^2)$$

Coefficients A, B, C_1, C_2 were computed along curves γ_4 using a special computer program. It appeared that along curves γ_4 shown in Fig.1, $A^2 + B^2 > 0$ the coefficient C_1 vanishes only at point P_1 ($e \simeq 0{,}985$, $\mu \approx 0{,}492$), changing there its sign.

If $C_1 \neq 0$, then in $\{\varepsilon, \delta : C_1\varphi_2(0, 0, \varepsilon, \delta) < 0\}$ Eq.(3.10) has two real roots

$$b'(\varepsilon, \delta) = [-C_1^{-1}\varphi_2(0, 0, \varepsilon, \delta)\,(1 + o(1))]^{1/2}, \; b''(\varepsilon, \delta) = -b'(\varepsilon, \delta)$$

to which correspond 4π-periodic solutions of Eq.(1.1)

$$x^{(0)}(t) = x_*(t) + q_*[t, 0, b'(\varepsilon, \delta), \varepsilon, \delta] \quad x^{(2)}(t) = x_*(t) + q_*[t, 0, b''(\varepsilon, \delta), \varepsilon, \delta] = x^{(0)}(t + 2\pi)$$

Curve γ_4 is defined in the (ε, δ) plane by the equation $\varphi_2(0, 0, \varepsilon, \delta) = 0$ and is the curve of branching of solutions $x^{(k)}(t)$ $(k = 0, 1, \ldots)$. The domains of existence of these solutions in the neighborhood of curves γ_4 are indicated in Fig.1 by hatching. It can be shown /3/ that one more curve of branching of solutions $x^{(k)}(t)$ $(k = 0, 1, \ldots)$ different from γ_4 issues from point P_1 (curve γ shown in Fig.2 which is nominal). Curves γ_4 and γ are tangent to each other in point P_1. The number of roots of Eq.(3.10) in the domains bounded by these curves is indicated by numerals in Fig.2.

Consider now Eq.(3.14). As shown by numerical computations, along these curves γ_4 $C_1 + C_2 \neq 0$. In the domain $\{\varepsilon, \delta: (C_1 + C_2)\varphi_2(0, 0, \varepsilon, \delta) < 0\}$ Eq.(3.14) has two real roots

$$b_*'(\varepsilon, \delta) = [-(C_1 + C_2)^{-1}\varphi_2(0, 0, \varepsilon, \delta)(1 + o(1))]^{1/2}, b_*''(\varepsilon', \delta) = -b_*'(\varepsilon, \delta)$$

to which correspond 4π-periodic solutions

$$x_*^{(0)}(t) = x_*(t) + q_*[t, b_*'(\varepsilon, \delta), b_*'(\varepsilon, \delta), \varepsilon, \delta]$$
$$x_*^{(2)}(t) = x_*(t) + q_*[t, b_*''(\varepsilon, \delta), b_*''(\varepsilon, \delta), \varepsilon, \delta] = x_*^{(0)}(t + 2\pi)$$

Curve γ_4 is the branching curve of these solutions. For small $6\varepsilon, \delta$ the domains of existence of solutions $x_*^{(k)}$ $(k = 0, 1, \ldots)$ lie above the γ_4 curves (Fig.1).

5. 6π-periodic solutions of the second kind. Curves γ_3 and γ_6 can be the branching curves of these solutions. The behavior of 6π-periodic solutions of Eq.(1.1) in the neighborhood of curves γ_6 is similar to that of 6π-periodic solutions of the equation studied in /3/, which is not investigated here. Consider curves γ_3 on which $m = 3$, $n = 2, 4, 8, \ldots$. Functions φ_1 and φ_2 in (3.9) will be represented in the form (4.1) for $C_1 = C_2 = C$ (cf.(3.15)).

Coefficients A, B, C were computed along curves γ_3 shown in Fig.1, $A^2 + B^2 > 0$, coefficient C vanishes only at points P_2 $(e \approx 0{,}590,\ \mu \approx 0{,}897)$ and P_3 $(e \approx 0{,}322,\ \mu \approx 2.01)$, where they change their sign. The branching of roots of Eqs.(3.10) and (3.11) along curves γ_3 is similar to that of roots of Eq.(3.10) on curves γ_4 investigated in Sect.4. The domains of existence of 6π-periodic solutions $x^{(k)}$, $\bar{x}^{(k)}$ $(k = 0, 1, \ldots)$ in the neighborhood of γ_3 curves are shown hatched in Fig.1.

6. Numerical computations of 4π-periodic solutions. Computed solutions of boundary value problems (1.4) and (1.5) with $m = 2$ which for $e \ll 1$ coincide with solutions $x_{1/2}^{(0)}, x_{1/2}^{(2)}$ and $\bar{x}_{1/2}^{(0)}, \bar{x}_{1/2}^{(2)}$, respectively are shown in Figs.3 and 4. The dependence of initial conditions of these solutions on e for various values of μ, and the subdivision of the (e, μ) plane in regions containing the same number of solutions. There are k solutions in the region denoted by E_k, while region E_0 is not indicated. The curves that delineate this subdivision are called branching curves.

Let us consider solutions of problem (1.4) when $m = 2$ (Fig.3). The solution that satisfies the inequality $x^{\cdot}(0) > 0$ $(x^{\cdot}(0) < 0)$ when $e = 0$ coincides with solution $x_{1/2}^{(0)}$ $(x_{1/2}^{(2)})$ if $e \ll 1$. These solutions form surface $S_{1/2}$ in the space $R^3(x^{\cdot}(0), e, \mu)$. The curves in Fig.3 can be interpreted as follows. Those in the plane $(e, x^{\cdot}(0))$ represent intersections of surface $S_{1/2}$ and planes $\mu = \text{const}$. The curves in the (e, μ) plane are othogonal projections on the (e, μ) plane of curves belonging to surface $S_{1/2}$ at whose points the plane tangent to $S_{1/2}$ is parallel to the $x^{\cdot}(0)$-axis. It can be shown that surface $S_{n/2}$ $(n = 1, 3, 5 \ldots)$ intersects surface S along the Γ_4 curve issuing from point $(0, 0, n^2/4)$. The odd 4π-periodic solutions of the second kind $x^{(0)}, x^{(2)}$ described in Sects.3 and 4 are solutions of the boundary value problem (1.4) with $m = 2$, whose initial velocities lie on surface $S_{n/2}$ in the neighborhood of curve $\Gamma_4 = S_{n/2} \cap S$. The respective curve of branching of these solutions is represented by γ_4. The solutions shown in Fig.3 have also the branching curve γ issuing from point P_1 (see Sect.4). Two different curves lying on surface $S_{1/2}$ are simultaneously projected on curve γ.

Surface $S_{n/2}$ has the following properties. Let $x = X(t, \alpha, e, \mu)$ be a solution of Eq.(1.1) with initial conditions $X(0, \alpha, e, \mu) = 0$, $X^{\cdot}(0, \alpha, e, \mu) = \alpha$. If point $Q_0 = (\alpha, e, \mu) \in S_{n/2}$, then also point $Q_1 = (X(2\pi, \alpha, e, \mu), e, \mu) \in S_{n/2}$. And when also $Q_0 \overline{\in} S$, then points Q_0 and Q_1 lie on $S_{n/2}$ on the opposite sides of S. One of these points corresponds to the solution continued from $x_{n/2}^{(0)}$, and the other to that continued from $x_{n/2}^{(2)}$. By virtue of the indicated property the curves on surface $S_{n/2}$ either coincide with Γ_4 (on which solutions continued from $x_{n/2}^{(0)}, x_{n/2}^{(2)}$ merge and degenerate into 2π-periodic), or exist in pairs at points where the plane tangent to $S_{n/2}$ is parallel to the $x^{\cdot}(0)$-axis. The projections of such curves on the (e, μ) plane do not coincide.

Let us consider solutions of the boundary value problem (1.5) with $m = 2$ (Fig.4). The solution that satisfies the inequality $x(\pi/2) > 0$ $(x(\pi/2) < 0)$ when $e = 0$ is the same as solution $\bar{x}_{1/2}^{(0)}$ $(\bar{x}_{1/2}^{(2)})$ when $e \ll 1$. To these solutions corresponds surface $S_{1/2}' \subset R^3(x(\pi/2), e, \mu)$. Curves γ_4 are the branching curves of solutions of the boundary value problem (1.5) with $m = 2$ (see Sects.3 and 4). Solutions represented in Fig.4 have also the branching curve $\bar{\gamma}$ whose origin is unrelated to curve γ_4. The projected image of surface $S_{1/2}'$ on the (e, μ) plane has a singularity of the type of accumulation at points which in such mapping become point $P_4(e \simeq 0{,}884,\ \mu = 0{,}894) \in \bar{\gamma}$ (Fig.4).

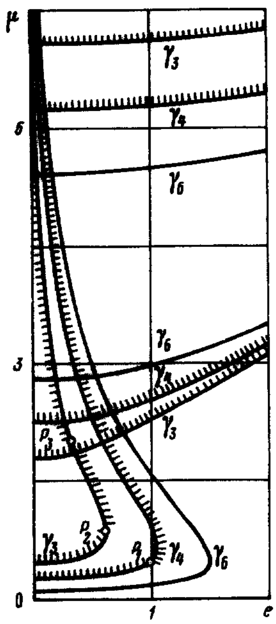

Fig.1

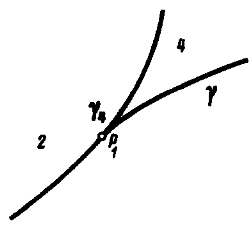

Fig.2

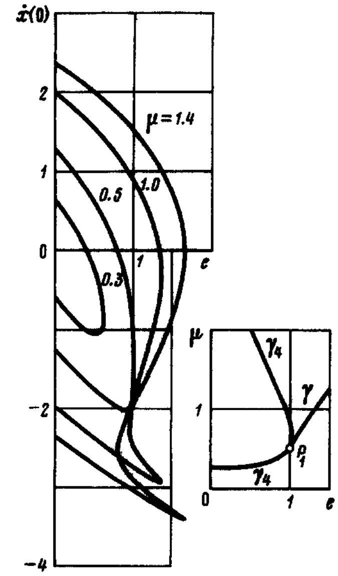

Fig.3

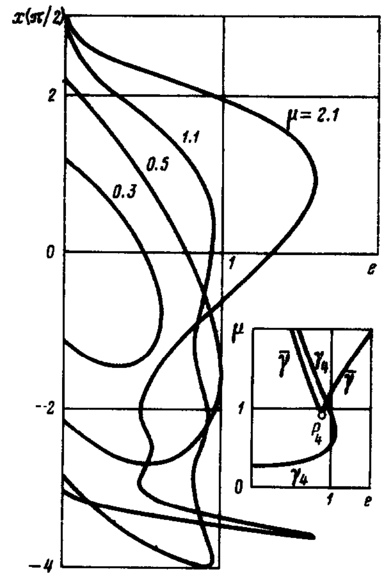

Fig.4

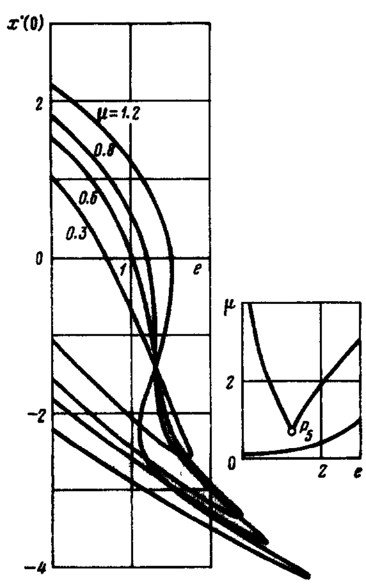

Fig.5

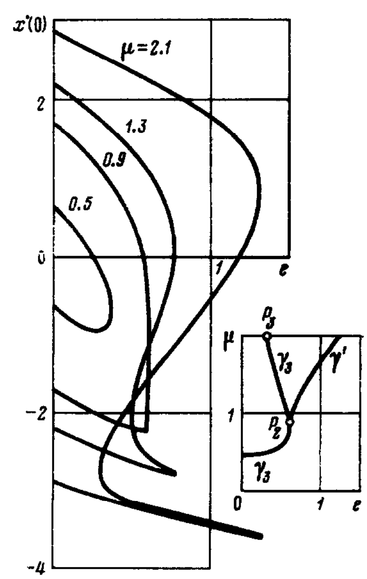

Fig.6

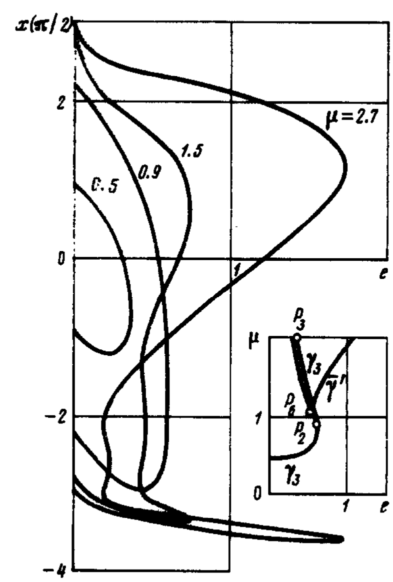

Fig.7

7. Numerical computations of 6π-periodic solutions. Solutions of the boundary value problems (1.3)—(1.5) with $m=3$ which are the same as solutions $x_{1/3}^{(0)}, x_{1/3}^{(3)}$ (Fig.5), $x_{2/3}^{(0)}, x_{2/3}^{(3)}$ (Fig.6), and $\bar{x}_{2/3}^{(0)}, \bar{x}_{2/3}^{(3)}$ (Fig.7) when $e \ll 1$ are shown in Figs.5—7. The intension of these figures is similar to that of Figs.3 and 4.

Let us consider solutions of the boundary value problem (1.3) when $m=3$ (Fig.5). The solution that satisfies the condition $x^{\cdot}(0)>0$ $(x^{\cdot}(0)<0)$ when $e=0$ is the same as solution $x_{1/3}^{(0)}$ $(x_{1/3}^{(3)})$ when $e \ll 1$. To these solutions corresponds surface $S_{1/3} \subset R^3(x^{\cdot}(0), e, \mu)$, and curve $S \cap S_{1/3}$ is now curve Γ_6 issuing from point $(0, 0, 1/9)$. The projected image of surface $S_{1/3}$ on the (e, μ) plane has at point P_5 $(e \simeq 1{,}267, \mu \approx 0{,}698)$ (Fig.5) a singularity of the accumulation type.

Let us pass to the solution of problem (1.4) when $m=3$ (Fig.6). The solution in which $e=0$ $x^{\cdot}(0)>0$ $(x^{\cdot}(0)<0)$ is the same as solution $x_{2/3}^{(0)}$ $(x_{2/3}^{(3)})$ when $e \ll 1$. These solutions are defined by surface $S_{2/3} \subset R^3(x^{\cdot}(0), e, \mu)$. Surface $S_{n/3}$ $(n=2, 4, 8, \ldots)$ intersects surface S along curve Γ_3 issuing from point $(0, 0, n^2/9)$. The odd 6π-periodic solutions of the second kind $x^{(0)}, x^{(3)}$ described in Sects.3 and 5 are solutions of the boundary value problem (1.4) when $m=3$ whose initial velocities lie on surface $S_{n/3}$ in the neighborhood of curve $\Gamma_3 = S \cap S_{n/3}$. The respective branching curve of these solutions is γ_3.

Surface $S_{n/3}$ has the following property. If $X(t, \alpha, e, \mu)$ is the function determined in Sect.6 and point $Q_0 = (\alpha, e, \mu) \in S_{n/3}$, then also point $Q_1 = (-X(3\pi, \alpha, e, \mu), e, \mu) \in S_{n/3}$. And, when in addition $Q_0 \overline{\in} S$, then points Q_0 and Q_1 lie on $S_{n/3}$ on opposite sides of S. One of these points corresponds to the solution continued from $x_{n/3}^{(0)}$ and the other to that continued from $x_{n/3}^{(3)}$. By virtue of the indicated property the curves on surface $S_{n/3}$ either coincide with curve Γ_3 or exist in pairs at points at which a plane tangent to $S_{n/3}$ is parallel to axis $x^{\cdot}(0)$. One such pair projected on curve γ' (Fig.6) issuing from point P_2 (see Sect. 5) was found on surface $S_{2/3}$. A branching curve also issuing from point P_3 lies to the left of curve γ_3 and runs in the direction of increasing μ. However this branching curve and the γ_3 curve are indistinguishable in the scale of Fig.6.

Let us consider solutions of the boundary value problem (1.5) (Fig.7) when $m=3$. The solution which for $e=0$ $x(\pi/2)>0$ $(x(\pi/2)<0)$ is the same as solution $\bar{x}_{2/3}^{(0)}$ $(\bar{x}_{2/3}^{(3)})$ when $e \ll 1$. These solutions are defined by surface $S_{2/3}' \subset R^3(x(\pi/2), e, \mu)$. The projection of curve $S' \cap S_{n/3}$ on the (e, μ) plane is represented by curve γ_3 issuing from point $(0, n^2/9)$ and is the branching curve of solution of the boundary value problem (1.5) when $m=3$.

The solutions illustrated in Fig.7 have two more branching curves, one of which joins points P_2 and P_3 and lies to the right of curve γ_3, from which it is indistiguishable in Fig.7. The other branching curve (curve $\bar{\gamma}'$ in Fig.7) is similar to curve $\bar{\gamma}$ in Fig.4. The projected image of surface $S_{2/3}'$ on the (e, μ) plane has a singularity of the accumulation type at points which become point P_6 $(e \approx 0{,}548, \mu \approx 1{,}029)$.

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